A CHARACTERIZATION OF THE CLASSICAL ORTHOGONAL POLYNOMIALS

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The classical orthogonal polynomials (Q_n) can be specificated as the Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$ $(\alpha, \beta > -1)$, the generalized Laguerre polynomials $L_n^s(t)$ $(s > -1)$, and finally as the Hermite polynomials $H_n(t)$. Their weight functions $t \mapsto w(t)$ on an interval of orthogonality (a, b) satisfy the differential equation

$$\frac{d}{dt}(A(t)w(t)) = B(t)w(t),$$

where the functions $t \mapsto A(t)$ and $t \mapsto B(t)$ are defined as in Table 1.

Table 1. The Classification of the Classical Orthogonal Polynomials

(a,b)	$w(t)$	$A(t)$	$B(t)$	λ_n	$Q_n(t)$
$(-1,1)$	$(1-t)^\alpha(1+t)^\beta$	$1-t^2$	$\beta-\alpha-(\alpha+\beta+2)t$	$n(n+\alpha+\beta+1)$	$P_n^{(\alpha,\beta)}(t)$
$(0,+\infty)$	$t^s e^{-t}$	t	$s+1-t$	n	$L_n^s(t)$
$(-\infty,+\infty)$	e^{-t^2}	1	$-2t$	$2n$	$H_n(t)$

The classical orthogonal polynomial $t \mapsto Q_n(t)$ is a particular solution of the following differential equation of the second order

$$L[y] = A(t)y'' + B(t)y' + \lambda_n y = 0, \tag{1}$$

where λ_n is given in the above table.

Let $(f, g) = \int_a^b f(t)g(t)w(t)\,dt$ and $\|f\|^2 = (f, f)$, and let $\mathcal{P}_n$ be the set of all algebraic polynomials of degree at most n. Similarly to the well-known Landau inequality [5] for continuously–differentiable functions and other generalizations (see, for example, [1–4] and [6–8]), in this short note we state the following characterization of the classical orthogonal polynomials.

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Theorem. *For all* $P_n \in \mathcal{P}_n$ *the inequality*

$$(2\lambda_n + B'(0))\|\sqrt{A}P_n'\|^2 \le \lambda_n^2\|P_n\|^2 + \|AP_n''\|^2 \tag{2}$$

holds, with equality if only if $P_n(t) = cQ_n(t)$*, where* Q_n *is the classical orthogonal polynomial of degree* n *orthogonal to all polynomials of degree* $\le n-1$ *with respect to the weight function* $t \mapsto w(t)$ *on* (a,b)*, and* c *is an arbitrary real constant. The* λ_n*,* $A(t)$ *and* $B(t)$ *are given in Table* 1*.*

Proof. Using (1) we have

$$\begin{aligned}\|L[P_n]\|^2 = \|AP_n''\|^2 + \|BP_n'\|^2 + \lambda_n^2\|P_n\|^2 \\ + 2(AP_n'', BP_n') + 2\lambda_n(AP_n'', P_n) + 2\lambda_n(BP_n', P_n).\end{aligned}$$

A simple application of integration by parts gives

$$2(AP_n'', BP_n') = -B'(0)\|\sqrt{A}P_n'\|^2 - \|BP_n'\|^2$$

and

$$\|\sqrt{A}P_n'\|^2 = -(AP_n'', P_n) - (BP_n', P_n).$$

Then, we find

$$\|L[P_n]\|^2 = \|AP_n''\|^2 - B'(0)\|\sqrt{A}P_n'\|^2 + \lambda_n^2\|P_n\|^2 - 2\lambda_n\|\sqrt{A}P_n'\|^2.$$

Since $\|L[P_n]\| \ge 0$, we obtain (2).

It is easy to see that the equality case is given by $P_n(t) = cQ_n(t)$. Namely, the polynomial solution of the equation (1) is only $cQ_n(t)$, where c is a constant. □

Now, we give the special cases.

First, for $w(t) = e^{-t^2}$ on $(-\infty, +\infty)$, the inequality (2) reduces to Varma's result [9]:

$$\|P_n'\|^2 \le \frac{1}{2(2n-1)}\|P_n''\|^2 + \frac{2n^2}{2n-1}\|P_n\|^2.$$

In the generalized Laguerre case, the inequality (2) becomes

$$\|\sqrt{t}P_n'\|^2 \le \frac{n^2}{2n-1}\|P_n\|^2 + \frac{1}{2n-1}\|tP_n''\|^2,$$

where $w(t) = t^s e^{-t}$ on $(0, +\infty)$.

In the Jacobi case $\big(A(t) = 1-t^2,\ w(t) = (1-t)^\alpha(1+t)^\beta$ on $(-1,1)\big)$ the inequality (2) reduces to the following inequality

$$\begin{aligned}\big((2n-1)(\alpha+\beta) + 2(n^2+n-1)\big)\|\sqrt{1-t^2}P_n'\|^2 \\ \le n^2(n+\alpha+\beta+1)^2\|P_n\|^2 + \|(1-t^2)P_n''\|^2.\end{aligned}$$

In the simplest case, when $\alpha = \beta = 0$ (Legendre case), we obtain

$$\|\sqrt{1-t^2}P_n'\|^2 \le \frac{n^2(n+1)^2}{2(n^2+n-1)}\|P_n\|^2 + \frac{1}{2(n^2+n-1)}\|(1-t^2)P_n''\|^2.$$

In Chebyshev case ($\alpha = \beta = -1/2$), we get

$$\|\sqrt{1-t^2}P_n'\|^2 \le \frac{n^4}{2n^2-1}\|P_n\|^2 + \frac{1}{2n^2-1}\|(1-t^2)P_n''\|^2,$$

where $\|f\|^2 = \int_{-1}^{1}(1-t^2)^{-1/2}f(t)^2\,dt$.

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