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Weighted Interpolation of Functions with Isolated Singularities

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We consider the weighted interpolation of functions with isolated singularities on bounded and unbounded intervals. The cases in the weighted L^p-norm ($1 \leq p < +\infty$) and also in the uniform weighted norm are considered. Numerical examples are included.

1. Introduction

The weighted polynomial approximation of continuous functions or of smooth functions with singular derivatives at some isolated points is of a theoretical interest and often it proves to be useful in many applications. For example, such functions occur as solutions of integral equations with discontinuous right hand side. While there exists a wide literature about the polynomial approximation of functions with singularities in the endpoints, the case of singular functions with singularities at isolated points inside the interval have been studied only recently ([1], [7], [8]).

The inner singularities add new difficulties and require a more careful examination of the behaviour of the approximating polynomial around these singularities.

In this paper we propose some interpolation processes for approximating such functions in $[-1, 1]$. Some kind of necessary and sufficient conditions are given for the convergence of such processes and the corresponding errors are estimated in the uniform norm.

Also, we consider the case of functions which are defined almost everywhere on the real semiaxis, are not bounded at 0 and $+\infty$, and are not continuous at some inner points. In this context we show that it sufficies to interpolate a "finite section" of these functions, making economies on the computation and obtaining the optimal error estimates.

A lot of numerical examples show the theoretical results with particular attention to the behaviour of the error and of the weighted Lebesgue function.

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The paper is organized as follows. The basic facts on weighted approximation of functions with isolated singularities are studied in Section 2. Interpolation processes on bounded and unbounded intervals are considered in Section 3 and the main results are stated there. Numerical examples are treated in Section 4, and the proofs of the theorems from Section 3 are given in Section 5.

2. Basic Facts

Let $u(x) = v^{\gamma,\delta}(x)|x-t_0|^\theta$ be a generalized Jacobi weight with $\theta, \gamma, \delta > 0$, $|t_0| < 1$, and $A := (-1, t_0) \cup (t_0, 1)$. We denote by $C^0(A)$ the set of continuous functions in A and introduce the space

$$L_u^\infty = \Big\{ f \in C^0(A) \colon \lim_{\substack{x\to\pm1 \\ x\to t_0}} (uf)(x) = 0 \Big\},$$

equipped with the norm

$$\|uf\|_{L_u^\infty} = \|uf\|_\infty = \sup_{|x|\le 1} |(uf)(x)|.$$

The functions in L_u^∞ are in general unbounded in a neighborhood of ±1 and/or t_0. They can be smooth with singular derivatives at the aforementioned points. For the sake of simplicity we have introduced only one interior point t_0, but a finite number of points of the same kind can be considered, whereas in this case obvious modifications in the notations are needed.

For smoother functions in A we define the following Sobolev-type space

$$W_r^\infty = \Big\{ f \in L_u^\infty \colon \|f^{(r)}\varphi^r u\|_\infty < +\infty \Big\}, \quad r \ge 1, \quad \varphi(x) = \sqrt{1-x^2}.$$

In order to characterize the functions in L_u^∞ it is useful to introduce the following modulus of continuity. Setting

$$\|\cdot\|_{L_u^\infty(\mathcal{B})} = \|\cdot\|_{\mathcal{B}}, \quad \mathcal{B} \subset A,$$

we define

$$\Omega_\varphi^r(f,t)_u = \sup_{0<h\le t} \|u\Delta_{h\varphi} f\|_{I_h},$$

where

$$\Delta_{h\varphi} f(x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^k f\left[x + \left(\frac{r}{2} - k\right) h\sqrt{1-x^2}\right]$$

and

$$I_h = (-1 + 4r^2h^2, t_0 - 4rh) \cup (t_0 + 4rh, 1 - 4r^2h^2).$$

Then, for any $f \in L^\infty_u$ the modulus of continuity ω^r_φ is defined as

$$\omega^r_\varphi(f,t)_u = \Omega^r_\varphi(f,t)_u + \sum_{k=0}^{2} \inf_{\deg q_k \le r} \|u(f-q_k)\|_{I_k(t)}, \tag{2.1}$$

where

$$I_0 = [-1, -1+4r^2t^2], \quad I_1 = [t_0 - 4rt, t_0 + 4rt]\,, \quad I_2 = [1-4r^2t^2, 1],$$

and t is "small" (say $t < t_0$).

By $\omega^r_\varphi(f,t)_u$ we can characterize the functions in L^∞_u in the following sense

$$f \in L^\infty_u \iff \lim_{t\to 0} \omega^r_\varphi(f,t)_u = 0.$$

Moreover, setting $E_n(f)_u = \inf_{p \in \mathcal{P}_n} \|u(f-p)\|$, it can be shown that (see [8])

$$E_n(f)_u \le \mathcal{C}\omega^r_\varphi(f, 1/n)_u \tag{2.2}$$

for any $f \in L^\infty_u$ and for some positive constant $\mathcal{C}$ independent of n and f (shortly $\mathcal{C} \ne \mathcal{C}(n,f)$).

For smoother functions in $A = (-1, t_0) \cup (t_0, 1)$, $E_n(f)_u$ can be estimated as follows. Namely, setting

$$g(t) = \sup_{0<h\le t} h^r \|f^{(r)}\varphi^r u\|_{I_h} \tag{2.3}$$

and

$$e(t) = \inf_{\deg q \le r-1} \|(f-q)|t_0 - \cdot|^\theta\|_{[t_0-4rt, t_0+4rt]}, \tag{2.4}$$

we have

$$E_n(f)_u \le \mathcal{C}\left[\int_0^{1/n} \frac{g(t)}{t}\,dt + \int_0^{2/n} \frac{e(t)}{t}\,dt\right]. \tag{2.5}$$

Notice that if $g(t) + e(t) \sim t^\lambda$, $0 < \lambda < r$, then also $\omega^r_\varphi(f,t)_u \sim t^\lambda$ (see [1]).

In this paper we consider also functions defined in $\mathbb{R}^+ = (0, \infty)$ and singular in some interior point (say t_0). We take the Laguerre weight $w_{2\gamma}(x) = x^{2\gamma}\exp(-x)$, and set

$$u(x) = \sqrt{w_{2\gamma}(x)}\,|x - t_0|^\theta, \qquad \theta, \gamma > 0.$$

Let C^0_u be a collection of all continuous functions in $\tilde{A} := \mathbb{R}^+ \backslash \{t_0\}$. Then, as in the case of finite interval, we define the space

$$L^\infty_u = \left\{ f \in \tilde{A}\colon \lim_{\substack{x\to 0\\ x\to t_0\\ x\to+\infty}} (uf)(x) = 0 \right\},$$

equipped with the norm

$$\|fu\|_{L^\infty_u} = \|fu\|_\infty = \sup_{x\geq 0} |(uf)(x)|.$$

For smoother functions we introduce the Sobolev-type space

$$\tilde{W}^\infty_r = \left\{ f \in L^\infty_u \colon \|f^{(r)}\varphi^r u\|_\infty < +\infty \right\}, \quad r \geq 0, \quad \varphi(x) = \sqrt{x}.$$

In order to introduce an appropriate modulus of continuity, at first we introduce the main part

$$\Omega^r_\varphi(f,t)_u = \sup_{0<h\leq t} \|u\Delta_{h\varphi} f\|_{I^*_h},$$

where $\Delta_{h\varphi}$ was defined before with $\varphi(x) = \sqrt{x}$, and

$$I^*_h = \left[4r^2h^2, \mathcal{C}/h^2\right] \setminus (t_0 - 4rh, t_0 + 4rh)$$

for some fixed $\mathcal{C} > 0$.

The modulus of continuity of $f \in L^\infty_u$ is defined as

$$\begin{aligned} \omega^r_\varphi(f,t)_u \quad = \quad & \Omega^r_\varphi(f,t)_u + \inf_{\deg q \leq r} \|u(f-q)\|_{[0,4rt^2]} \\ & + \inf_{\deg q \leq r-1} \|u(f-q)\|_{[t_0-4rt, t_0+4rt]} \\ & + \inf_{\deg q \leq r} \|u(f-q)\|_{[\mathcal{C}/t^2, +\infty)}. \end{aligned} \tag{2.6}$$

As was shown in ([2])

$$f \in L^\infty_u \iff \lim_{t\to 0} \omega^r_\varphi(f,t)_u = 0.$$

Besides, setting $E_n(f)_u = \inf_{p\in \mathcal{P}_n} \|u(f-p)\|$, we have

$$E_n(f)_u \leq \mathcal{C}\omega^r_\varphi(f, 1/n)_u \tag{2.7}$$

for any $f \in L^\infty_u$ and $\mathcal{C} \neq \mathcal{C}(n,f)$.

As in the case of a finite interval, we can define

$$g^*(t) = \sup_{0<h\leq t} h^r \|f^{(r)}\varphi v\|_{I^*_h}$$

and

$$e^*(t) = \inf_{\deg q \leq r} \|v(f-q)\|_{[t_0-4rt, t_0+4rt]},$$

in order to obtain the following estimate

$$E_n(f)_v \leq \mathcal{C}\left[\int_0^{2/\sqrt{n}} \frac{g^*(t)}{t}\,dt + \int_0^{2/\sqrt{n}} \frac{e^*(t)}{t}\,dt\right]. \tag{2.8}$$

The quantity $\omega_\varphi^r(f,t)_v$ can be replaced by the right-hand side of (2.8), when the last one is of the order t^λ, $0<\lambda<r$ (see [2]).

In the sequel, it shall need the following lemma from [1]:

Lemma 2.1. *Suppose that f is a function such that $f^{(r-1)}$ is absolutely continuous in $[t_0-a,t_0+a]\backslash\{t_0\}$ and $\|\sigma f^{(r)}\|_{[t_0-a,t_0+a]}<+\infty$ for a small a (say $a<\bar{a}$) and $\sigma(x)=|x-t_0|^\gamma$, $\gamma>0$. Let $|t|<a$. If $\gamma>r$ then*

$$\|\sigma f\|_{[-t,t]}\le \mathcal{C}t^r\left[\|\sigma f^{(r)}\|_{[t_0-a,t_0+a]}+\|\sigma f\|_{[t_0-a,t_0+a]}\right].$$

However, if $\gamma\le r$ ($\gamma\notin\mathbb{Z}$) and $f^{(r-[\gamma]-1)}(t_0)$ exists, there are polynomials $p\in\mathcal{P}_{r-[\gamma]-1}$ such that

$$\|\sigma(f-p)\|_{[-t,t]}\le \mathcal{C}t^r\left[\|\sigma f^{(r)}\|_{[t_0-a,t_0+a]}+\|\sigma f\|_{[t_0-a,t_0+a]}\right],$$

with $\mathcal{C}\ne\mathcal{C}(f,t)$.

3. Interpolation Processes on Bounded and Unbounded Intervals

Let $f\in L_u^\infty$ and $u(x)=v^{\gamma,\delta}(x)|x-t_0|^\theta$ ($\gamma,\delta,\theta\ge 0$, $|t_0|<1$). If we want to approximate the function f by a Lagrange interpolating polynomial, the point t_0 cannot be an interpolation knot, and therefore we use the following procedure.

Let $w(x)=v^{\alpha,\beta}(x)|x-t_0|^\eta$ be another generalized Jacobi weight and let $\{p_n(w)\}$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients. Let us denote by $x_1<x_2<\cdots<x_n$ the zeros of $p_n(w)$. Let x_c be the closest zero to t_0, i.e., $|x_c-t_0|=\min\limits_k|x_k-t_0|$, and let $q_s\in\mathcal{P}_s$ be such that

$$\|(f-q_s)|\cdot-t_0|^\theta\|_{\left[t_0-\frac{a}{n},\,t_0+\frac{a}{n}\right]}\le 2\inf_{\deg q\le s}\|(f-q_s)|\cdot-t_0|^\theta\|_{\left[t_0-\frac{a}{n},\,t_0+\frac{a}{n}\right]},$$

with a fixed $a>0$. Let $\psi\in C^\infty(\mathbb{R})$ be a nondecreasing function such that

$$\psi(x)=\begin{cases}0, & x\le 0\\ 1, & x\ge 1.\end{cases}$$

Using ψ we define the functions

$$\psi_1(x)=\psi\left(\frac{x-x_{c-2}}{x_{c-1}-x_{c-2}}\right),\quad \psi_2(x)=\psi\left(\frac{x-x_{c+1}}{x_{c+2}-x_{c+1}}\right),\tag{3.1}$$

and

$$F=F_{t_0}=(1-\psi_1)f+(1-\psi_2)\psi_1 q_s+\psi_2 f.$$

It follows from the definition that

$$F = \begin{cases} f & \text{in } [-1, x_{c-2}] \cup [x_{c+2}, 1], \\ q_s & \text{in } [x_{c-1}, x_{c+1}], \\ (1-\psi_1)f + (1-\psi_2)\psi_1 q_s & \text{in } [x_{c-2}, x_{c-1}], \\ (1-\psi_2)q_s + \psi_2 f & \text{in } [x_{c+1}, x_{c+2}]. \end{cases} \tag{3.2}$$

Next we interpolate the function F at the zeros $x_1 < x_2 < \cdots < x_n$ of $p_n(w)$ and we denote by $\tilde{L}_n(w, F)$ the corresponding Lagrange polynomial. Recalling (3.2) we have

$$\begin{aligned} \tilde{L}_n(w, F, x) &= \sum_{k=1}^{n} l_k(x) F(x_k) \\ &= \sum_{k \neq c,\, c\pm 1}^{n} l_k(x) f(x_k) + \sum_{k=c-1}^{c+1} l_k(x) q_s(x_k), \end{aligned} \tag{3.3}$$

where

$$l_k(x) = \frac{p_n(w, x)}{p_n'(w, x_k)(x - x_k)}.$$

Denoting by $\|\cdot\|_p$ the usual L^p norm $(1 \le p < \infty)$, we can state the following result for $\tilde{L}_n(w)$.

Theorem 3.1. *Let* $f \in L_u^\infty$, $u(x) = v^{\gamma,\delta}(x)|x - t_0|^\theta$ $(\gamma, \delta \ge 0,\ 0 \le \theta < 1,\ |t_0| < 1)$, *and* $1 \le p < +\infty$. *Then*

$$\|u\tilde{L}_m(w, F)\|_p \le \mathcal{C}\|uF\|_\infty, \quad \mathcal{C} \neq \mathcal{C}(m, F), \tag{3.4}$$

if and only if

$$\frac{u}{\sqrt{w\varphi}} \in L^p \quad \text{and} \quad \frac{\sqrt{w\varphi}}{u} \in L^1, \quad \varphi(x) = \sqrt{1 - x^2}. \tag{3.5}$$

Moreover,

$$\|u\tilde{L}_n(w, F)\|_\infty \le \mathcal{C}\|uF\|_\infty \log n, \quad \mathcal{C} \neq \mathcal{C}(n, F), \tag{3.6}$$

if and only if

$$\frac{u}{\sqrt{w\varphi}} \in L^\infty \quad \text{and} \quad \left(\frac{\sqrt{w\varphi}}{u} \in L^1 \quad \text{or} \quad \begin{matrix} \gamma = \dfrac{\alpha}{2} + \dfrac{5}{4} \\ \delta = \dfrac{\beta}{2} + \dfrac{5}{4} \\ \theta = \dfrac{\eta}{2} + 1 \end{matrix} \right). \tag{3.7}$$

Corollary 3.1. *Suppose that f and u satisfy the conditions in Theorem* 3.1 *and let*

$$A_n(f) = E_{n-1}(f)_u + \inf_{\deg q \le s} \|u(f-q)\|_{[t_0-\frac{a}{n}, t_0+\frac{a}{n}]}.$$

Then for $1 \le p < +\infty$ we have

$$\|u[f - \tilde{L}_n(w,F)]\|_p \le \mathcal{C} A_n(f), \quad \mathcal{C} \ne \mathcal{C}(n,F), \tag{3.8}$$

if and only if (3.5) *holds. Moreover,*

$$\|u[f - \tilde{L}_n(w,F)]\|_\infty \le \mathcal{C} \log n \, A_n(f), \quad \mathcal{C} \ne \mathcal{C}(n,F), \tag{3.9}$$

if and only if (3.7) *holds.*

Theorem 3.1 and Corollary 3.1 will be proved in Section 5. Now, we give some remarks.

For $\theta = 0$, Theorem 3.1 follows directly from [11, Theorem 1], but for $\theta > 0$ some nontrivial difficulties appear in the proof.

We deduce from the definition of ω_φ^r that

$$A_n(f) \le \mathcal{C} \omega_\varphi^r (f, 1/n)_u .$$

If the function f is smooth "around" the singularity, by using (2.5), $A_n(f)$ can be estimated as follows

$$A_n(f) \le \mathcal{C} \left[\int_0^{2/n} \frac{g(t)}{t} \, dt + \int_0^{2/n} \frac{e(t)}{t} \, dt \right].$$

For example, if $f(x) = \operatorname{sgn}(x)$, then $g(t) = 0$,

$$e(t) = \inf_{\deg q \le r-1} \|u(\operatorname{sgn} - q)\|_{[t_0 - 4rt, t_0 + 4rt]} \le \mathcal{C} \| |t - \cdot|^\theta \|_{[t_0 - 4rt, t_0 + 4rt]} \sim t^\theta,$$

and $A_n(f) \le \mathcal{C} n^{-\theta}$.

In particular, if $f^{(r-1)}(t_0)$ exists and $\|f^{(r)} \varphi^r u\| < +\infty$, with $r \ge 1$, then using Lemma 2.1, we can obtain an estimate for $A_n(f)$ of the following form

$$A_n(f) \le \frac{\mathcal{C}}{n^r} \left(\|f^{(r)} \varphi^r u\| + \|uf\| \right).$$

Moreover, if the above assumptions on f are satisfied, then we can set in (3.3) $q_s(x_k) = f(x_k)$. The conditions (3.5) and (3.7) can be expressed as follows

$$\begin{aligned} \frac{\alpha}{2} - \frac{1}{4} - \frac{1}{p} \; &< \; \gamma \; < \; \frac{\alpha}{2} + \frac{5}{4}, \\ \frac{\beta}{2} - \frac{1}{4} - \frac{1}{p} \; &< \; \delta \; < \; \frac{\beta}{2} + \frac{5}{4}, \\ \frac{\eta}{2} - \frac{1}{p} \; &< \; \theta \; < \; \frac{\eta}{2} + 1 \end{aligned} \tag{3.10}$$

and

$$\begin{array}{rcccl} \frac{\alpha}{2}-\frac{1}{4} & \le & \gamma & \le & \frac{\alpha}{2}+\frac{5}{4}, \\ \frac{\beta}{2}-\frac{1}{4} & \le & \delta & \le & \frac{\beta}{2}+\frac{5}{4}, \\ \frac{\eta}{2} & \le & \theta & \le & \frac{\eta}{2}+1, \end{array} \tag{3.11}$$

respectively.

It is not difficult to see that the (strong) assumption $0 < \theta < 1$ is not required from the conditions (3.5) and (3.7), but from the presence of the weight u in the norm of the function (see the proof of Theorem 3.1). However, if $\theta \ge 1$ in the weight u, a slight modification can be made in the previous Lagrange polynomial $\tilde{L}_n(w, F)$. In fact, it is sufficient to interpolate the function $F = F_{t_0}$ at the zeros of $p_{n+1}(w, x)/(x - x_c)$, where $x_c = x_{n+1,c}$, defining the following interpolation process

$$\begin{aligned} L_n^*(w, F, x) &= \sum_{k=1,\, k\ne c}^{n} l_k(x) \frac{x_k - x_c}{x - x_c} F(x_k) \\ &= \sum_{k\ne c,\, c\pm 1}^{n} l_k(x) f(x_k) \frac{x_k - x_c}{x - x_c} + \sum_{k=c-1,\, k\ne c}^{c+1} l_k(x) q_s(x_k) \frac{x_k - x_c}{x - x_c} \end{aligned}$$

where l_k is defined like in (3.3) with $n+1$ instead of n. For this last polynomial the following theorem, complementary in some sense to Theorem 3.1, holds.

Theorem 3.2. *Let f and u be as in Theorem 3.1 and $1 \le p < +\infty$. Then there exists a positive constant $\mathcal{C} \ne \mathcal{C}(n, F)$ such that*

$$\|uL_n^*(w, F)\|_p \le \mathcal{C}\|uF\|_\infty \tag{3.12}$$

if and only if

$$\frac{u}{|\cdot - t_0|\sqrt{w\varphi}} \in L^p \quad \text{and} \quad \frac{|\cdot - t_0|\sqrt{w\varphi}}{u} \in L^1. \tag{3.13}$$

Moreover, for some positive constant $\mathcal{C} \ne \mathcal{C}(n, f)$ we have

$$\|uL_n^*(w, f)\|_\infty \le \mathcal{C}\|uf\|_\infty \log n \tag{3.14}$$

if and only if

$$\frac{u}{|\cdot - t_0|\sqrt{w\varphi}} \in L^\infty \quad \text{and} \quad \left(\frac{|\cdot - t_0|\sqrt{w\varphi}}{u} \in L^1 \quad \text{or} \quad \begin{array}{c} \gamma = \frac{\alpha}{2}+\frac{5}{4} \\ \delta = \frac{\beta}{2}+\frac{5}{4} \\ \theta = \frac{\eta}{2}+1 \end{array} \right). \tag{3.15}$$

We will omit the proof of this theorem since it is very similar to that of Theorem 3.1. Of course, (3.8) and (3.9) of Corollary 3.1 hold again, setting $L_n^*(w)$ instead of $\tilde{L}_n(w)$, and if (3.13) replaces (3.5), and (3.15) replaces (3.7).

From (3.15) it follows

$$1 + \frac{\eta}{2} \le \theta \le 2 + \frac{\eta}{2},$$

i.e., $\theta > 1/2$ and Theorem 3.2 is not true for $\theta \le 1/2$. Therefore, the interpolation processes $\{\tilde{L}_n(w,F)\}$ and $\{L_n^*(w,F)\}$ are complementary and they can approximate every function in L_u^∞. However, $\tilde{L}_n(w)$ and $L_n^*(w)$ use the zeros of the generalized Jacobi polynomial and their construction (except some special cases) requires a high computational cost, since until now only few properties of these polynomials are known. To overcome this problem we propose a third procedure which uses the zeros of Jacobi polynomials and which replaces $L_n^*(w)$ (not $\tilde{L}_n(w)$!).

Indeed, following an idea from [3], let $v^{\alpha,\beta}$ be the Jacobi weight and let $\{p_n(v^{\alpha,\beta})\}$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients. Given $\nu \in \mathbb{N}$, let $x_1 < x_2 < \cdots < x_{n+\nu}$ be the zeros of $p_{n+\nu}(v^{\alpha,\beta})$ and let us denote by x_c the zero of $p_{n+\nu}(v^{\alpha,\beta})$ which is closest to t_0, i.e., $|x_c - t_0| = \min_k |x_k - t_0|$. Moreover, let $y_i < \cdots < x_c < \cdots < y_\nu$ be ν zeros of $p_{n+\nu}(v^{\alpha,\beta})$ of type $x_{c\pm(i-1)}$. We set $\pi(x) = \prod_{i=1}^{\nu}(x - y_i)$. Finally, let $L_n(v^{\alpha,\beta}, f)$ be the Lagrange polynomial interpolating $f \in L_u^\infty$ at the zeros of $p_{n+\nu}(v^{\alpha,\beta}, x)/\pi(x)$, i.e.,

$$L_n(v^{\alpha,\beta}, f, x) = \sum_{x_k \in \mathcal{B}} \frac{p_{n+\nu}(v^{\alpha,\beta}, x) f(x_k)}{\pi(x)} \frac{\pi(x_k)}{p'_{n+\nu}(v^{\alpha,\beta}, x_k)(x - x_k)},$$

where $\mathcal{B} = \{y_1, \ldots, y_\nu\}$.

Now, we are able to state the following theorem which is similar to the previous one.

Theorem 3.3. *Let f and u be as in Theorem 3.1 and $1 \le p < +\infty$. Then there exists a positive constant $\mathcal{C} \ne \mathcal{C}(n, F)$ such that*

$$\|uL_n(v^{\alpha,\beta}, f)\|_p \le \mathcal{C}\|uf\|_\infty \tag{3.16}$$

if and only if

$$\frac{u}{|\cdot - t_0|^\nu \sqrt{v^{\alpha,\beta}\varphi}} \in L^p \quad \text{and} \quad \frac{|\cdot - t_0|^\nu \sqrt{v^{\alpha,\beta}\varphi}}{u} \in L^1. \tag{3.17}$$

Moreover, for some positive constant $\mathcal{C} \ne \mathcal{C}(n, f)$ we have

$$\|uL_n(v^{\alpha,\beta}, f)\|_\infty \le \mathcal{C}\|uf\|_\infty \log n \tag{3.18}$$

if and only if

$$\frac{u}{|\cdot - t_0|^\nu \sqrt{v^{\alpha,\beta}\varphi}} \in L^\infty \text{ and } \left(\frac{|\cdot - t_0|^\nu \sqrt{v^{\alpha,\beta}\varphi}}{u} \in L^1 \text{ or } \begin{array}{c} \gamma = \dfrac{\alpha}{2} + \dfrac{5}{4} \\ \delta = \dfrac{\beta}{2} + \dfrac{5}{4} \\ \nu = \theta - 1 \end{array} \right). \tag{3.19}$$

From (3.19) it follows that $\theta - 1 \leq \nu \leq \theta$ and therefore, since $\nu \geq 1$, this implies $\theta \geq 1$. Theorem 3.2 can be replaced in numerical applications by the last theorem (but not by Theorem 3.1). Notice that (3.16) is equivalent to

$$\|u[f - L_n(v^{\alpha,\beta}, f)]\|_p \leq \mathcal{C}E_{n-1}(f)_u, \quad 1 \leq p < +\infty,$$

and (3.18) to

$$\|u[f - L_n(v^{\alpha,\beta}, f)]\|_\infty \leq \mathcal{C}E_{n-1}(f)_u \log n.$$

To simplify the notations we have assumed that f has only one singular point (i.e., the weight u has only one interior zero). In the case of two or more points, for instance if $u(x) = v^{\gamma,\delta}(x)|x - t_0|^{\theta_0}|x - t_1|^{\theta_1}$, we use the zeros of the generalized Jacobi polynomials orthogonal with respect to the weight

$$w(x) = v^{\alpha,\beta}(x)|x - t_0|^{\eta_0}|x - t_1|^{\eta_1}$$

and we have to construct a new function F by modifying the function f around the singularities t_0 and t_1. If we use Jacobi zeros, then we consider the zeros of $p_{n+\nu_1+\nu_2}(v^{\alpha,\beta})$ and interpolate f at the zeros of

$$\frac{p_{n+\nu_1+\nu_2}(v^{\alpha,\beta}, x)}{\pi_{\nu_1}(x)\pi_{\nu_2}(x)},$$

where π_{ν_1} and π_{ν_2} are defined as before.

Also, we consider functions $f \in L^\infty_v$, where

$$v(x) = \sqrt{w_{2\gamma}(x)}\,|x - t_0|^\eta, \quad w_{2\gamma}(x) = x^{2\gamma}e^{-x}, \quad t_0 > 0,\ \eta - \gamma > 0.$$

For such functions we are not able to establish the complete results obtained in the case of bounded intervals. In fact, very little is known about the orthogonal polynomials with respect to the weights like $|x - t_0|^\lambda e^{-x}$ till now and, moreover, the behaviour of the weighted L_p-norm of the Lagrange polynomials based on the Laguerre zeros is not much clear.

Here we propose the following procedure.

Let w_α be the Laguerre weight, $w_\alpha(x) = x^\alpha e^{-x}$, $\alpha > -1$, $x > 0$. Let $\{P_n(w)\}$ be the corresponding system of orthonormal polynomials with positive leading coefficients and let $x_1 < \cdots < x_{n+\nu}$, $\nu \geq 1$, be the zeros of $P_{n+\nu}(w_\alpha)$ where $x_c := x_{n+\nu,c}$ is one of the closest zeros to t_0. We denote by $y_1 < \cdots <$

$x_c < \cdots < y_\nu$ the zeros of $P_{n+\nu}(w_\alpha)$ of the kind $x_{c\pm(i-1)}$ and set $\pi(x) = \prod_{i=1}^{\nu}(x-y_i)$. Moreover, let $j := j(n)$ be such that $x_j = \min\{x_k \geq 4\theta(n+\nu)\}$, $0 < \theta < 1$. Using the above introduced function ψ, we define $\psi_j(x) := \psi\left(\frac{x-x_j}{x_{j+1}-x_j}\right)$ and $f_j := (1-\psi_j)f$. Finally, we denote by $L_{n+1}(w_\alpha, f_j)$ the Lagrange polynomial interpolating the function f_j at the zeros of the polynomial

$$\frac{4(n+\nu)-x}{\pi(x)}\, P_{n+\nu}(w_\alpha, x).$$

Since $f_j = f$ in $(0, x_j)$ and $f_j = 0$ in $[x_{j+1}, +\infty)$, we can write

$$L_{n+1}(w_\alpha, f_j, x) = \sum_{\substack{k=1\\ x_k \notin \mathcal{B}}}^{j} l_k^*(x) f(x_k),$$

where $\mathcal{B} = \{y_1, \ldots, y_\nu\}$ and

$$l_k^*(x) = \frac{4(n+\nu)-x}{4(n+\nu)-x_k}\,\frac{P_{n+\nu}(w_\alpha, x)}{\pi(x)}\,\frac{\pi(x_k)}{P'_{n+\nu}(w_\alpha, x_k)(x-x_k)}.$$

Now, we state the following result.

Theorem 3.4. *Let $f \in L_v^\infty$, $v(x) = x^\gamma |x-t_0|^\eta e^{-x/2}$, with $\gamma > 0$ and $\eta \geq 1$. Then, with $M = [\frac{\theta}{1+\theta} n]$, $0 < \theta < 1$, we have*

$$\|v[f - L_{n+1}(w_\alpha, f_j)]\|_\infty \leq \mathcal{C}\left[E_M(f)_v \log n + e^{-An}\|vf\|_\infty\right]$$

if and only if

$$\frac{\alpha}{2} + \frac{1}{4} \leq \gamma \leq \frac{\alpha}{2} + \frac{5}{4} \quad \text{and} \quad \eta - 1 \leq \nu \leq \eta, \tag{3.20}$$

where $\mathcal{C}$ and A are positive constants independent of n and f.

Notice that Theorem 3.4 still holds true if $j = n$, but the “truncation” introduced by $L_{n+1}(w_\alpha, f_j)$ allows us to neglect the computation of $\mathcal{O}(n)$ terms of the sum and it can be useful in the applications. Finally, we note that an approximation of functions defined on the whole real axis and with some singular points can be also obtained using a similar argument, but we omit details.

4. Examples

In this section we consider a few examples in order to illustrate the previous theoretical results, especially ones given in Theorem 3.3 (for $p = +\infty$) and

Theorem 3.4. All computations were performed in MATHEMATICA 3.0 system, using the standard machine precision known as double precision (m.p. $\approx 2.22 \times 10^{-16}$).

For the interval $[-1, 1]$ we take the weight u as in Theorem 3.3, i.e., $u(x) = v^{\gamma,\delta}(x)|x - t_0|^\theta$, where $v^{\gamma,\delta}(x) = (1-x)^\gamma(1+x)^\delta$ (Jacobi weight) and $\gamma, \delta \geq 0$, $\theta \geq 1$. The interpolation nodes are zeros of the Jacobi polynomial $p_{n+\nu}(v^{\alpha,\beta}, x)$, excluding ν of them which are closest to the singular point $x = t_0$. We also present the corresponding *weighted Lebesgue function*,

$$\Lambda_n(u, x) = u(x) \sum_{k=1}^{n} \frac{l_{n,k}(x)}{u(x_k)}, \tag{4.1}$$

where the interpolation nodes are denoted as x_k $(k = 1, \dots, n)$ and $l_{n,k}(x)$ are the fundamental Lagrange polynomials. The behaviour of the Lebesgue function plays an important role in interpolation processes (cf. [4]).

For the interval $[0, +\infty)$ we take the "space" weight $v(x) = x^\gamma e^{-x/2}|x - t_0|^\eta$, with $\gamma \geq 0$ and $\eta \geq 1$. The interpolation nodes are the zeros of the generalized Laguerre polynomial $P_{n+\nu}(w_\alpha, x)$ $(w_\alpha(x) = x^\alpha e^{-x})$, excluding ν of them, which are the closest to the singular point $x = t_0$, and adding the node $4(n + \nu)$. According to Theorem 3.4, a "truncation" of the Lagrange sum can be used, taking only j terms, where $j := j(n)$ is determined by $x_j = \min\{x_k \geq 4\theta(n+\nu)\}$ and $0 < \theta \leq 1$.

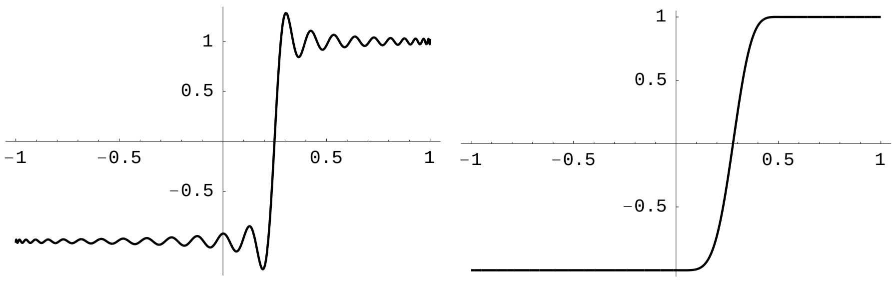


Figure 4.1: Non-weighted (left) and weighted (right) Lagrange polynomial for $f(x) = \operatorname{sgn}(x - 1/4)$ and $n = 50$ nodes

Example 4.1. As a first example we consider the simple function $f(x) = \operatorname{sgn}(x - 1/4)$ which has a singularity at the point $x = t_0 = 1/4$. Because of that a non-weighted Lagrange interpolation is bad. The case of such interpolation at $n = 50$ Chebyshev nodes is displayed in Figure 4.1 (left).

Since the function f is regular at ± 1, according to Theorem 3.3, we put $\gamma = \delta = 0$. As a weight function (Jacobi weight $v^{\alpha,\beta}$) we can take the Chebyshev weight of the first kind,

$$w(x) = v^{-1/2,-1/2}(x) = \frac{1}{\sqrt{1-x^2}},$$

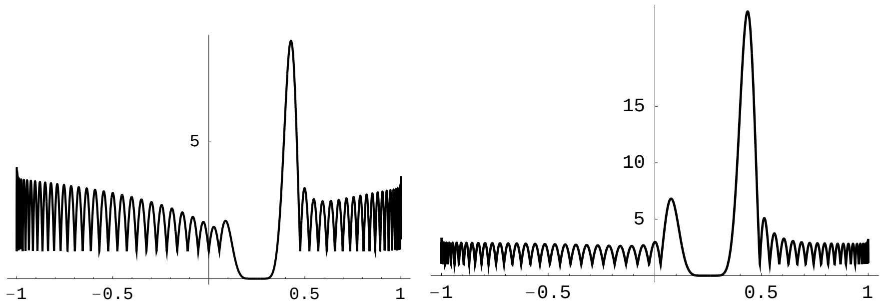


Figure 4.2: The weighted Lebesgue function for $n = 50$ and $\nu = 7$ (left) and $\nu = 8$ (right)

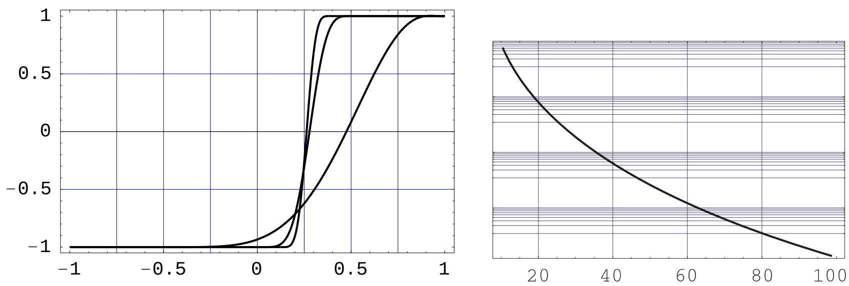


Figure 4.3: The weighted Lagrange polynomial for $n = 10, 50$, and 100 nodes (left) and the uniform norm of the weighted error for $n \leq 100$ (right)

because $\alpha = \beta = -1/2$ satisfy the conditions

$$\frac{\alpha}{2} + \frac{1}{4} \leq \gamma \leq \frac{\alpha}{2} + \frac{5}{4} \quad \text{and} \quad \frac{\beta}{2} + \frac{1}{4} \leq \delta \leq \frac{\beta}{2} + \frac{5}{4}\,. \tag{4.2}$$

Then, the interpolation nodes x_k $(k = 1, \ldots, n)$ will be the zeros of $T_{n+\nu}(x)$, excluding ν $(\theta - 1 \leq \nu \leq \theta)$ of them, which are closest to the point $t_0 = 1/4$.

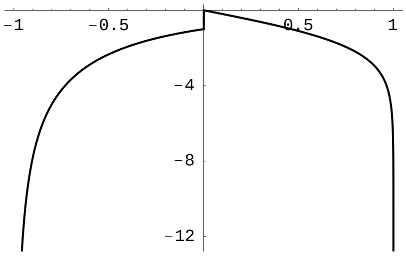
Taking $\theta = 8$ we extract $\nu = 7$ or $\nu = 8$ zeros of $T_{n+7}(x)$ or $T_{n+8}(x)$, respectively. The weighted Lebesgue functions in these cases are given in Figure 4.2. We take in our calculation $\nu = 7$, because this case gives slightly better results then the second one. The corresponding weighted Lagrange polynomial in this case for $n = 50$ is displayed in Figure 4.1 (right). The cases for $n = 10$, $n = 20$, and $n = 100$ are shown in Figure 4.3 (left).

The uniform norm of the weighted error, $\|u[f - L_n(v^{\alpha,\beta}, f)]\|_\infty$, for $n \leq 100$ is presented in Figure 4.3 (right) as a linear-log plot.

Example 4.2. Consider the function f defined by

$$f(x) = \begin{cases} -\dfrac{e^{-x}}{\sqrt{1+x}}, & \text{for } x < 0, \\ \log \dfrac{1-x}{1+x}, & \text{for } x > 0. \end{cases}$$

Besides the end-point singularities, a singularity at $x = t_0 = 0$ exists with a jump equal to $\lim_{x\to 0+} f(x) - \lim_{x\to 0-} f(x) = 1$ (see Figure 4.4 (left)).



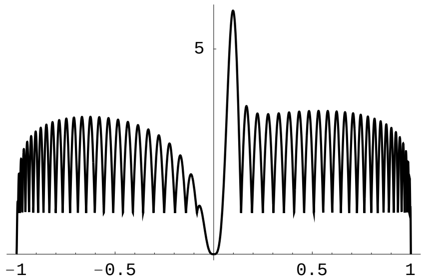
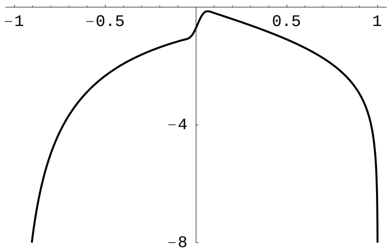


Figure 4.4: The graphics $x \mapsto f(x)$ (left) and the weighted Lebesgue function $x \mapsto \Lambda_n(u, x)$ (right) for $n = 50$ nodes

We put $u(x) = (1-x)^{3/2}(1+x)^{5/2}|x|^{7/2}$ and $\alpha = 3/2$, $\beta = 7/2$, so that the conditions (4.2) are satisfied. Taking $\theta = 7/2$, we must extract $\nu = 3$ nodes from the set of all zeros of the Jacobi polynomial $p_{n+\nu}(v^{3/2,7/2}, x)$. The corresponding weighted Lebesgue function (4.1) for $n = 50$ is presented in Figure 4.4 (right).



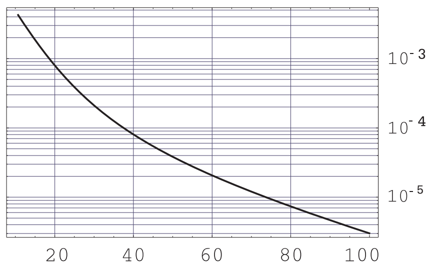


Figure 4.5: The weighted Lagrange polynomial for $n = 100$ nodes (left) and the uniform norm of the weighted error for $n \leq 100$ (right)

The uniform norm of the weighted error for $n \leq 100$ is displayed in Figure 4.5 (right). The weighted Lagrange polynomial $L_{100}(v^{3/2,7/2}, f, x)$ is given on the left-hand side of the same figure. Notice that for a small n, e.g. $n = 10$, this polynomial is a bad approximation to f (see Figure 4.6 (left)). On the

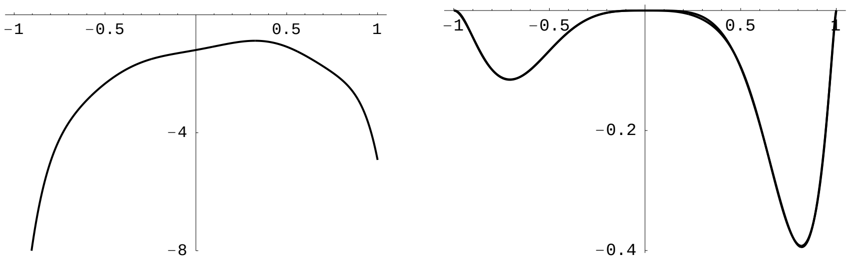


Figure 4.6: The weighted Lagrange polynomial $x \mapsto L_{10}(v^{3/2,7/2}, f, x)$ (left) and the corresponding functions $x \mapsto u(x)L_{10}(v^{3/2,7/2}, f, x)$ and $x \mapsto u(x)f(x)$ (right)

other side, we can see that $u(x)L_{10}(v^{3/2,7/2}, f, x)$ is very close to $u(x)f(x)$, i.e., $\|u[f - L_{10}(v^{3/2,7/2}, f)]\|_\infty \approx 4.5 \times 10^{-3}$.

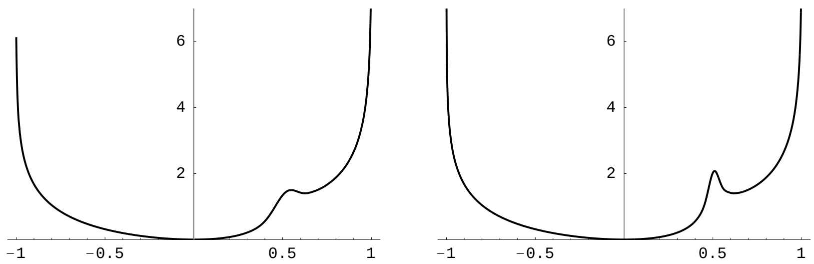


Figure 4.7: The weighted Lagrange polynomial for $n = 50$ (left) and $n = 100$ nodes (right)

Example 4.3. Let

$$f(x) = \frac{1}{\sqrt{|\sin(x - 1/2)|}} \log \frac{1}{1 - x^2} .$$

As we can see

$$\lim_{x \to \pm 1} f(x) = +\infty \quad \text{and} \quad \lim_{x \to 1/2} f(x) = +\infty.$$

We take $\gamma = \delta = 3/2$ and $\theta = 5/2$, i.e., $u(x) = (1 - x^2)^{3/2}|x - 1/2|^{5/2}$, and $\alpha = \beta = 3/2$. Notice that $\nu = 2$ in this case. The weighted Lagrange polynomials for $n = 50$ and $n = 100$ are displayed in Figure 4.7. Figure 4.8 shows these polynomials and the original function $x \mapsto f(x)$ in the interval $(-1, 1)$ (left) and locally for $x \in (0.4, 0.6)$ (right).

In Figure 4.9 we present the graphics of $x \mapsto u(x)L_{10}(v^{3/2,3/2}, f, x)$ and $x \mapsto u(x)f(x)$ (left), as well as the corresponding Lebesgue function for $n = 50$

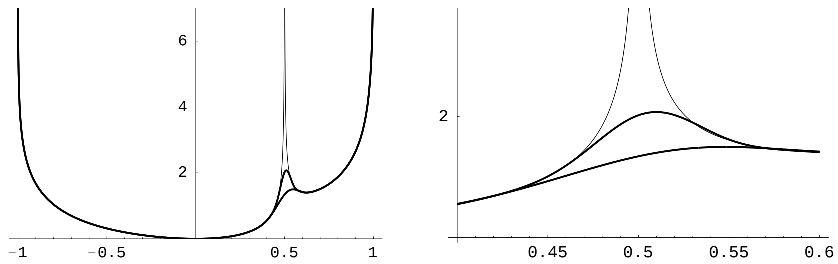


Figure 4.8: The graphics $x \mapsto L_n(v^{3/2,3/2}, f, x)$ $(n = 50, 100)$ and $x \mapsto f(x)$ in $(-1, 1)$ (left) and $(0.4, 0.6)$ (right)

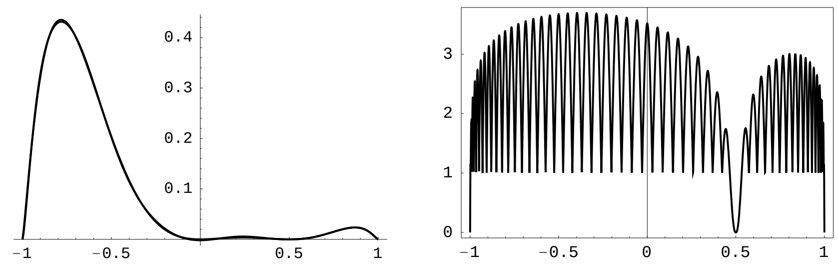


Figure 4.9: The graphics of $x \mapsto u(x)L_{10}(v^{3/2,3/2}, f, x)$ and $x \mapsto u(x)f(x)$ (left) and the Lebesgue function $x \mapsto \Lambda_n(u, x)$ for $n = 50$ nodes (right)

nodes (right). We mention also that the uniform norm of the weighted error $\|u[f - L_n(v^{3/2,3/2}, f)]\|_\infty$ is equal to 7.49×10^{-3}, 6.02×10^{-5}, and 7.73×10^{-6}, for $n = 10, 50$, and 100, respectively.

Example 4.4. Let $f(x) = e^{-x}|\sin 5(x - 1/2)|$. This function is continuous for $x \in [-1, 1]$, but there are three "critical points" in $(-1, 1)$:

$$t_0 = \frac{1}{2} - \frac{2\pi}{5}, \quad t_1 = \frac{1}{2} - \frac{\pi}{2}, \quad t_2 = \frac{1}{2},$$

in which the function f is not differentiable (see Figure 4.10).

A direct application of the Lagrange interpolation with Chebyshev nodes gives the results in Table 4.1. In the second column of this table we give the uniform norm of the corresponding errors $\bar{e}_n(x) = f(x) - \bar{L}_n(v^{-1/2,-1/2}, x)$ for $n = 10(10)100$. Numbers in parentheses indicate the decimal exponents.

According to Theorem 3.3 and the corresponding comments regarding this theorem, we put $\gamma = \delta = 0$, $\theta_0 = \theta_1 = \theta_2 = 7/2$, so that

$$u(x) = |x - t_0|^{7/2}|x - t_1|^{7/2}|x - t_2|^{7/2}.$$

This allows to take the Chebyshev nodes ($\alpha = \beta = -1/2$) as zeros of $T_{n+9}(x)$ and to extract nine points (three of them in the neighborhood of each point t_k,

Number of nodes n	Standard interpolation $\|\bar{e}_n\|_\infty$	$\|u\bar{e}_n\|_\infty$	Weighted interpolation $\|ue_n\|_\infty$
10	2.60(−1)	8.18(−2)	2.92(−5)
20	2.00(−1)	8.31(−3)	5.51(−6)
30	8.14(−2)	2.16(−3)	6.02(−7)
40	1.47(−1)	5.26(−3)	3.08(−7)
50	9.72(−2)	2.45(−3)	1.07(−7)
60	6.07(−2)	2.54(−3)	3.47(−8)
70	5.96(−2)	1.12(−3)	2.77(−8)
80	4.67(−2)	1.19(−4)	1.50(−8)
90	3.89(−2)	1.70(−4)	9.10(−9)
100	3.94(−2)	5.40(−4)	6.46(−9)

Table 4.1: The uniform norm of the errors in the Lagrange interpolation

$k = 0, 1, 2$). The weighted Lebesgue function for such distribution of nodes is displayed in Figure 4.10 (right).

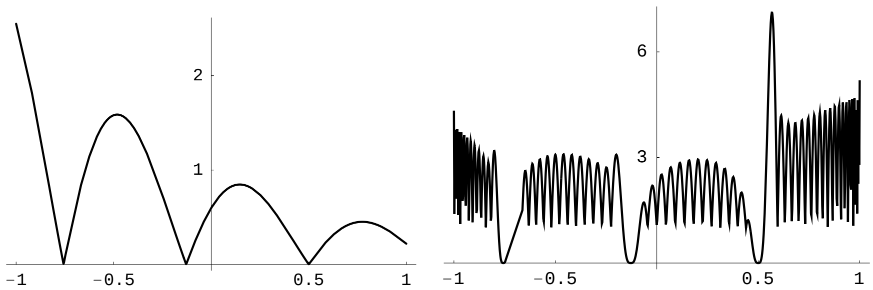


Figure 4.10: The graphics $x \mapsto f(x)$ (left) and the weighted Lebesgue function $x \mapsto \Lambda_n(u, x)$ (right) for $n = 60$

The uniform norm of the corresponding weighted error

$$u(x)e_n(x) = u(x)[f(x) - L_n(v^{-1/2,-1/2}, x)]$$

is presented in the last column of Table 4.1. In order to compare the errors in non-weighted and weighted interpolation, we also introduce an additional column in this table, with the uniform norm of the previous error of standard interpolation $\bar{e}_n(x)$ multiplied by $u(x)$. As we can see, the advantage of the weighted interpolation is evident.

In Figure 4.11 we give the graphics of the Lagrange interpolation polynomial $L_n(v^{-1/2,-1/2}, x)$ for $n = 60$ and the uniform norm $\|ue_n\|_\infty$ for $n \le 100$ (see also Table 4.1).

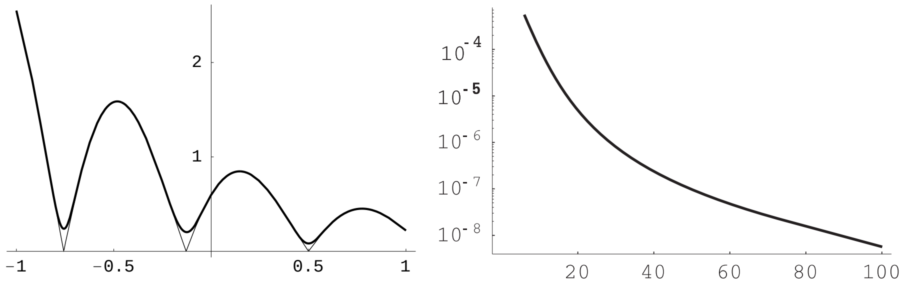


Figure 4.11: The Lagrange polynomial $x \mapsto L_{60}(v^{-1/2,-1/2}, x)$ (left) and the uniform norm of the weighted error $x \mapsto u(x)[f(x) - L_n(v^{-1/2,-1/2}, x)]$ (right) for $n \le 100$

Example 4.5. Consider the function f defined on $(0, +\infty)$ by

$$f(x) = \frac{e^{x/4}}{\sqrt{x}} \operatorname{sgn}(x - 10).$$

According to Theorem 3.4 we put $\gamma = 3/2$ and $\eta = 2$, i.e.,

$$v(x) = x^{3/2} e^{-x/2} |x - 10|^2.$$

The graphics of $x \mapsto f(x)$ and $x \mapsto v(x)f(x)$ are displayed in Figure 4.12.

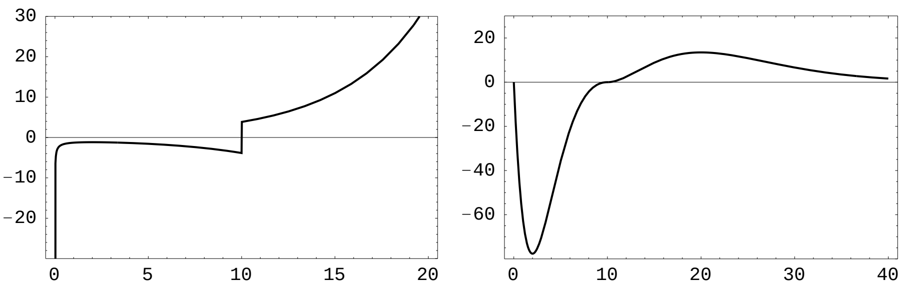


Figure 4.12: The graphics $x \mapsto f(x)$ (left) and $x \mapsto v(x)f(x)$ (right)

For the parameters α and ν which satisfy inequalities (3.20) we can take $\alpha = 5/2$ and $\nu = 1$. In this way, the weight w_α becomes the generalized Laguerre weight

$$w_{5/2}(x) = x^{5/2} e^{-x}, \qquad 0 \le x < +\infty.$$

In Figure 4.14 we present the graphic of the Lagrange polynomial $L_{n+1}(w_{5/2}, x)$ multiplied by the "space" weight $v(x)$ for $n = 10$, as well as the graphic of the corresponding weighted error.

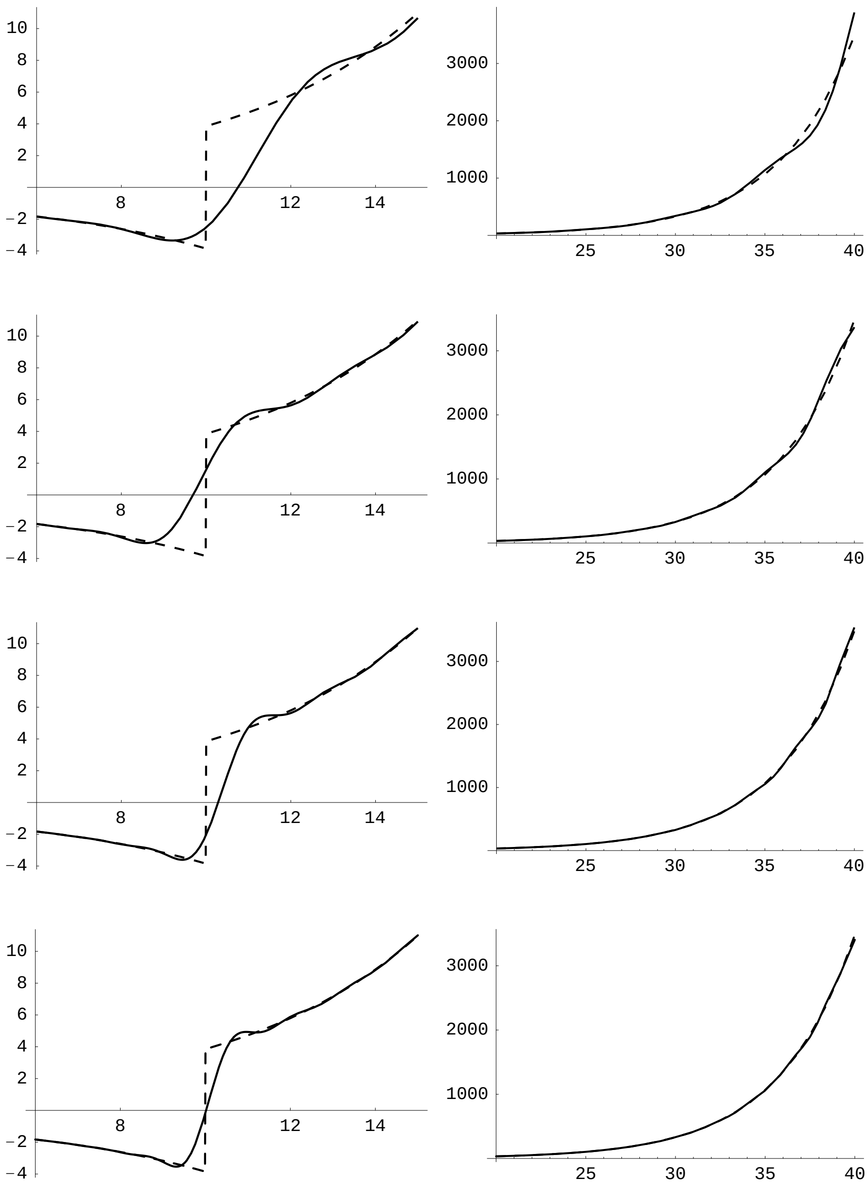


Figure 4.13: The graphics $x \mapsto L_{n+1}(w_{5/2}, x)$ (solid line) and $x \mapsto f(x)$ (broken line) on $[6, 15]$ (left) and $[20, 40]$ (right) for $n = 50$, $n = 100$, $n = 200$, and $n = 300$

Especially, it is interesting to consider the behaviour of the Lagrange polynomial $L_{n+1}(w_{5/2}, x)$ in a neighborhood of the singular point $x = 10$. Figure 4.13 shows the graphics of the Lagrange polynomial $x \mapsto L_{n+1}(w_{5/2}, x)$ and the function $x \mapsto f(x)$ for $x \in [6, 15]$, when $n = 50, 100, 200$, and 300. The behaviour of the interpolation polynomial in the interval $[20, 40]$ is also presented.

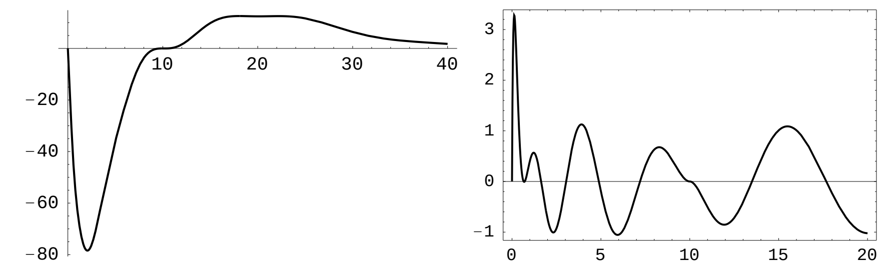


Figure 4.14: The function $x \mapsto v(x)L_{n+1}(w_{5/2}, x)$ (left) and the weighted error $x \mapsto v(x)[f(x) - L_{n+1}(w_{5/2}, x])$ (right) for $n = 10$

The graphics of the weighted Lebesgue function $x \mapsto \Lambda_{n+1}(x)$ in this case for $n = 10$ and $n = 50$ are displayed in Figure 4.15.

With a "truncation" of the Lagrange polynomial, i.e., taking only j terms, determined by $x_j = \min\{x_k \geq 4\theta(n + \nu)\}$ and $0 < \theta \leq 1$, the computations can be significantly reduced. The corresponding weighted Lebesgue function is denoted by $\Lambda_{n+1}^{(\theta)}(x)$. The cases for $n = 50$ with dropped nodes when $\theta = 1/2$ and $\theta = 1/4$ are presented in Figure 4.16.

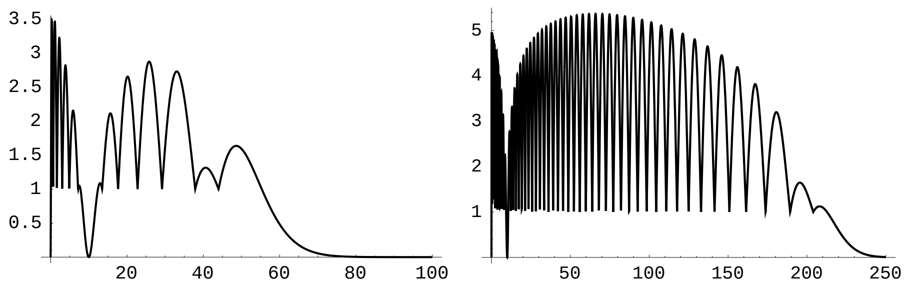
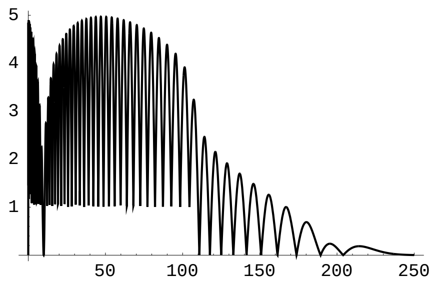


Figure 4.15: The weighted Lebesgue function $x \mapsto \Lambda_{n+1}(x)$ for $n = 10$ (left) and $n = 50$ (right)

As we can see, the corresponding weighted Lebesgue constants,

$$\Lambda_{n+1}^{(\theta)} = \max_{0 \leq x < +\infty} |\Lambda_{n+1}^{(\theta)}(x)|,$$

for $n = 50$ are almost the same when $\theta = 1$, $\theta = 1/2$, and $\theta = 1/4$. In other words, such a "truncation" in the weighted Lagrange polynomial does not change its numerical characteristics, but significantly reduced the computation.



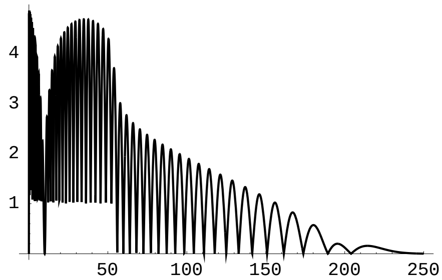


Figure 4.16: The weighted Lebesgue function $x \mapsto \Lambda_{n+1}^{(\theta)}(x)$ for $n = 50$ with dropped nodes: $\theta = 1/2$ (left) and $\theta = 1/4$ (right)

5. Proofs of the Statements

At first, we need to prove a preliminary lemma. Let

$$-1 = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = 1$$

with $x_k = \cos\theta_k$ and $n\,(\theta_{k-1} - \theta_k) \sim 1$. Here and in the sequel, if $A, B > 0$ are quantities depending on some parameters, we write $A \sim B$, if and only if there exist two positive constants M_1 and M_2, independent of the parameters A and B, such that

$$M_1 \le \left(\frac{A}{B}\right)^{\pm 1} \le M_2.$$

Set

$$\Gamma_n(x) := \sum_{k=1,k\neq d}^{n} \frac{v^{\mu,\nu}(x)\left(|x-t_0| + n^{-1}\right)^{\rho}}{v^{\mu,\nu}\,(x_k)\,(|x_k - t_0| + n^{-1})^{\rho}} \frac{\Delta x_k}{|x - x_k|}\,,$$

where $x_d = \min\limits_k |x_k - x|$, $\Delta x_k = x_{k+1} - x_k$, $\mu, \nu, \rho \in \mathbb{R}$.

In a similar way, let $y_1, \ldots, y_n$ be the zeros of the n-th Laguerre polynomial $p_n\,(w_\alpha)$ orthogonal on $(0, +\infty)$ with respect to the weight $w_\alpha(x) = x^\alpha e^{-x}$. Set

$$A_n(x) := \sum_{k=1,k\neq d}^{n} \frac{x^\sigma\,(|t_0 - x| + 1/\sqrt{n})^\tau}{y_k^\sigma\,(|t_0 - y_k| + 1/\sqrt{n})^\tau} \frac{\Delta y_{k-1}}{|x - y_k|}\,,$$

where $y_d = \min\limits_k |x - y_k|$, $\Delta y_{k-1} = y_k - y_{k-1}$ and $\sigma, \tau \in \mathbb{R}$.

Lemma 5.1. *Let $a \in \mathbb{R}^+$ be a fixed number. We have*

$$\sup_{|x|\le 1-a/n^2} \Gamma_n(x) \sim \log n \tag{5.1}$$

if and only if $0 < \mu,\ \nu, \rho < 1$. *Moreover,*

$$\sup_{a/n\le x\le 4n} A_n(x) \sim \log n \tag{5.2}$$

if and only if $0 < \sigma,\ \tau < 1$.

Proof. Let us prove (5.1). Since, for $x_k \neq x_d$, we have

$$\Gamma_n(x) = \sum_{x_k\le 0} + \sum_{x_k>0},$$

where in the first sum $1 - x_k \sim 1$ and in the second one $1 + x_k \sim 1$. Then, it will be sufficient to estimate separately

$$\Gamma_n' = \sum_{k=1,k\neq d}^{n} \left(\frac{1+x}{1+x_k}\right)^{\nu} \left(\frac{|x-t_0|+n^{-1}}{|x_k-t_0|+n^{-1}}\right)^{\rho} \frac{\Delta x_k}{|x-x_k|}$$

and

$$\Gamma_n'' = \sum_{k=1,k\neq d}^{n} \left(\frac{1-x}{1-x_k}\right)^{\nu} \left(\frac{|x-t_0|+n^{-1}}{|x_k-t_0|+n^{-1}}\right)^{\rho} \frac{\Delta x_k}{|x-x_k|}.$$

Let us consider Γ_n'. Let $\delta > 0$ be such that $\Delta = (t_0-\delta, t_0+\delta) \subset (-1,1)$. Then, $\Gamma_n' = \sum_{x_k\in\Delta} + \sum_{x_k\notin\Delta}$, $x_k \neq x_d$. In the first sum $1+x_k \sim 1$, and in the second one $|x_k - t_0| + n^{-1} \sim 1$. Since a similar decomposition holds also for Γ_n'', it is sufficient to estimate separately the next three sums

$$\sum_{k=1,k\neq d}^{n} \left(\frac{1-x}{1-x_k}\right)^{\mu} \frac{\Delta x_k}{|x-x_k|},$$

$$\sum_{k=1,k\neq d}^{n} \left(\frac{|x-t_0|+n^{-1}}{|x_k-t_0|+n^{-1}}\right)^{\rho} \frac{\Delta x_k}{|x-x_k|},$$

$$\sum_{k=1,k\neq d}^{n} \left(\frac{1+x}{1+x_k}\right)^{\mu} \frac{\Delta x_k}{|x-x_k|},$$

since $\mu, \nu, \rho > 0$. But, all these sums are equivalent to $\log n$, when $0 \le \mu, \nu, \rho \le 1$ (see [10]).

Moreover, if $\nu > 1$ we have

$$\begin{aligned}
\sup_{|x|\le 1-\frac{a}{n^2}} \Gamma_n(x) &\ge \Gamma_n(t_0/2) \\
&\ge \sum_{x_1\le x_k<0} \frac{v^{\mu,\nu}(t_0/2)\left(t_0/2+n^{-1}\right)^\rho}{v^{\mu,\nu}(x_k)\left(|t_0-x_k|+n^{-1}\right)^\rho} \frac{\Delta x_k}{|t_0/2-x_k|} \\
&> \frac{v^{\mu,\nu}(t_0/2)}{2}\left(\frac{t_0/2+n^{-1}}{2}\right)^\rho \sum_{x_1\le x_k<0} \frac{\Delta x_k}{(1+x_k)^\mu} \\
&\ge \mathcal{C}\int_{x_1}^{1/2} \frac{dt}{(1+t)^\nu} \\
&\sim n^{2(\nu-1)} > \log n,
\end{aligned}$$

and for $\nu < 0$,

$$\begin{aligned}
\sup_{|x|\le 1-\frac{a}{n^2}} \Gamma_n(x) &\ge \Gamma_n(x_1) \\
&> (1-x_1)^\mu (1+x_1)^\nu \left(|t_0-x_1|+n^{-1}\right)^\rho \\
&\times \sum_{0<x_k\le t_0/2} \frac{1}{(1-x_k)^\mu (1+x_k)^\nu \left(|t_0-x_k|+n^{-1}\right)^\rho} \frac{\Delta x_k}{|x_1-x_k|} \\
&\sim (1+x_1)^\nu \sum_{0<x_k\le t_0/2} \frac{\Delta x_k}{(1+x_k)^\nu} \\
&\sim (1+x_1)^\nu \int_0^{t_0/2} \frac{dt}{(1+t)^\nu} \\
&\sim n^{-2\nu} > \log n.
\end{aligned}$$

One can proceed in a similar way if $\mu < 0$ or $\mu > 1$.

Now, if $\rho > 1$, it is sufficient to evaluate Γ_n at $t_0/2$ in order to get

$$\begin{aligned}
\Gamma_n(t_0/2) &\ge \mathcal{C} \sum_{t_0-\delta<x_k<t_0+\delta} \frac{\Delta x_k}{\left(|t_0-x_k|+n^{-1}\right)^\rho} \\
&\ge \mathcal{C}\int_{t_0-\delta}^{t_0} \frac{dt}{\left[(t_0-t)+n^{-1}\right]^\rho} \sim n^{\rho-1}.
\end{aligned}$$

Finally, if $\rho < 0$ one has

$$\Gamma_n\left(t_0-n^{-1}\right) \ge n^{-\rho} \sum_{t_0+\delta/2<x_k<t_0+\delta} \frac{\Delta x_k}{\left(|t_0-x_k|+n^{-1}\right)^\rho} \sim n^{-\rho}$$

and the proof of (5.1) is complete.

We omit the proof of (5.2) because it is similar to the previous one (see [5, Lemma 4.1]). □

Proof of Theorem 3.1. Setting

$$A = \left(-1 + \frac{\mathcal{C}}{n^2}, t_0 - \frac{\mathcal{C}}{n}\right) \cup \left(t_0 + \frac{\mathcal{C}}{n}, 1 - \frac{\mathcal{C}}{n^2}\right),$$

for any fixed $\mathcal{C} > 0$, by Remez inequality we can write

$$\|u\widetilde{L}_n(w,F)\|_p \le \mathcal{C}\|u\widetilde{L}_n(w,F)\|_{L^p(A)}, \quad 1 \le p \le +\infty.$$

Putting $g(x) = \operatorname{sgn}(\widetilde{L}_n(w,F,x))|u(x)\widetilde{L}_n(w,F,x)|^{p-1}$ and

$$\begin{aligned} r(t) &= \int_A \frac{p_n(w,x) - p_n(w,t)}{x-t} u(x)g(x)\,dx \\ &= H\left(p_n(w)ug, t\right) - p_n(w,t)H(ug,t), \end{aligned}$$

where H denotes the Hilbert transform extended to A, we get

$$\|u\widetilde{L}_n(w,F)\|^p_{L^p(A)} = \sum_{k=1}^n \frac{F(x_k)\,u_n(x_k)}{p_n'(w,x_k)\,u_n(x_k)} r(x_k),$$

where $u_n(x) = v^{\gamma,\delta}(x)\left(|x-t_0| + n^{-1}\right)$.

Now, for $k \ne c$ $\left(|t_0 - x_c| = \min_k |t_0 - x_k|\right)$ we conclude that

$$u_n(x_k) \sim u(x_k).$$

For $x = c$ we have

$$\begin{aligned} F(x_c)\,u_n(x_c) &= q_s(x_c)\,u_n \\ &= q_s(x_{c-1})\,u_n(x_c) + u_n(x_c)\int_{x_{c-1}}^{x_c} q_s'(t)dt. \end{aligned}$$

Now, $u_n(x_c) \le u_n(x_{c-1}) \le cu(x_{c-1})$ and

$$|q_s(x_{c-1})\,u_n(x_c)| \le c\|uF\|_{[x_{c-1},x_c],\infty}.$$

Moreover, for $0 < \theta < 1$ we get

$$\begin{aligned} \left|u_n(x_c)\int_{x_{c-1}}^{x_c} q_s'(t)dt\right| &\le \frac{\mathcal{C}}{n^\theta}\int_{x_{c-1}}^{x_{c+1}} |q_s'(t)|\,|t_0 - t|^\theta \frac{dt}{|t_0 - t|^\theta} \\ &\le \frac{\mathcal{C}}{n^\theta}\left\| q_s' \,|t_0 - \cdot|^\theta \right\|_{[x_{c-1},x_{c+1}]} \int_{x_{c-1}}^{x_{c+1}} \frac{dt}{|t_0 - t|^\theta} \\ &\sim \frac{\mathcal{C}}{n^\theta}\cdot\frac{1}{n^{1-\theta}}\left\| q_s' \,|t_0 - \cdot|^\theta \right\|_{[x_{c-1},x_{c+1}]} \\ &\le \mathcal{C}\|uF\|, \end{aligned}$$

by using Markov inequality in $[x_{c-1}, x_{c+1}]$.

Then, for any $k = 1, \ldots, n$,

$$|u_n(x_k) F(x_k)| \le \|uF\|.$$

Furthermore (see [11])

$$\frac{1}{p_n'(w, x_k)} \sim \sqrt{w_n \varphi}\,(x_k)\,(x_{k+1} - x_k)\,,$$

where $w_n(x) = v^{\alpha,\beta}\left(|x - t_0| + n^{-1}\right)^{\eta}$. By using Marcinkiewicz inequality and Remez inequality, we get for $p \in [1, +\infty)$

$$\left\| u\widetilde{L}_n(w, F) \right\|_p \le \mathcal{C}\|uF\| \int_A \frac{\sqrt{w\varphi}(t)}{u(t)} |r(t)|\, dt,$$

where the integral exists by virtue of (3.5).

Furthermore, recalling the definition of $r(t)$, it holds

$$\begin{aligned}
\int_A \frac{\sqrt{w\varphi}(t)}{u(t)}|r(t)|dt \quad &\le \quad \int_A \frac{\sqrt{w\varphi}(t)}{u(t)} \left|H\left(p_n(w)gu, t\right)\right| dt \\
&+ \quad \int_A \frac{\sqrt{w\varphi}(t)}{u(t)} |p_n(w,t)|\, |H(gu,t)| dt =: I_1 + I_2.
\end{aligned}$$

Taking into account that $|p_n(w,t)| \le c/\sqrt{w\varphi}(t)$ for $t \in A$, and the equality

$$\int_A fHg = -\int_A gHf \qquad \text{for } f \in (L\log^+ L) \quad \text{and} \quad g \in L^\infty,$$

we have

$$I_1 \le \int_A \frac{u(t)}{\sqrt{w\varphi}(t)} g(t) \left| H\left(G_1 \frac{\sqrt{w\varphi}}{u}, t\right)\right| dt,$$

$$I_2 \le \int_A \frac{1}{u(t)} |H(gu,t)| dt \le \int_A g(t)u(t) \left| H\left(\frac{G_2}{u}, t\right)\right| dt,$$

where $G_1 = \operatorname{sgn} H(\ldots)$ and $G_2 = \operatorname{sgn} H(\ldots)$.

If $1 < p < +\infty$ we use [11, Lemma] and the estimate

$$I_1 + I_2 \le C\, \|L_n(w,f)u\|_p^{p-1}.$$

For $p = 1$, we have $|g(t)| \le 1$ and

$$I_1 \le \int_A H(G,t) dt + \int_A \frac{u(t)}{\sqrt{w\varphi}(t)} \left(\int_A \left| \frac{\frac{\sqrt{w\varphi}(x)}{u(x)} - \frac{\sqrt{w\varphi}(t)}{u(t)}}{x - t} \right| dx \right) dt.$$

Since
$$U(x)=\frac{\sqrt{w\varphi}(x)}{u(x)}$$
is a generalized Jacobi weight, the internal integral is dominated by a constant time the product of the factors of U having negative exponents. Then $I_1 \le \mathcal{C}$ since
$$\int_A H(G,t)dt \le \sqrt{2}\left(\int_A H^2(G,t)dt\right)^{1/2} \le 2.$$
For I_2 we use a similar argument. Thus (3.5) implies (3.4).

Let us prove that (3.4) implies (3.5). Now, from (3.4) it follows easily from [9, Theorem 2.2] that
$$\frac{u}{\sqrt{w\varphi}} \in L^p, \qquad 1 \le p < +\infty.$$
We have to prove that (3.4) implies that $\sqrt{w\varphi} \in L^1$, i.e. that $\alpha,\beta,\gamma,\delta,\theta,\eta$ satisfy (3.10). This can be done following [11, p. 688].

For θ and η, let us consider a function f with $|f(x)| \le 1$ such that $f(\pm 1)=0$, $f(x_k)=0$ for $x_k \le t_0-\delta$ and $x_k \ge t_0+\delta$, and $f(x_k)=\mathrm{sgn}\,(p_n'(w,x_k))$ if $x_k \in (t_0-\delta, t_0+\delta) \subset (-1,1)$. Then
$$\widetilde{L}_n(w,f,x) = p_n(w,x) \sum_{t_0-\delta \le x_k \le t_0+\delta} \frac{1}{|p_n'(w,x_k)|\,(x-x_k)\,u_n(x_k)}$$
and
$$\begin{aligned} |u(x)\widetilde{L}_n(w,f,x)| &\ge 2\,|p_n(w,x)u(x)| \sum_{t_0-\delta \le x_k \le t_0+\delta} \frac{1}{|p_n'(w,x_k)|\,u_n(x_k)} \\ &\sim |p_n(w,x)u(x)| \sum_{t_0-\delta \le x_k \le t_0+\delta} \frac{\sqrt{w\varphi}(x_k)}{u_n(x_k)}\Delta x_k \\ &\sim |p_n(w,x)u(x)| \sum_{t_0-\delta \le x_k \le t_0+\delta} \left(|x_k-t_0|+n^{-1}\right)^{\eta/2-\theta}\Delta x_k \\ &\sim |p_n(w,x)u(x)| \int_{t_0-\delta}^{t_0+\delta} \left(|t-t_0|+n^{-1}\right)^{\eta/2-\theta} dt. \end{aligned}$$
Then
$$\sup_n \|p_n(w)u\|_p \int_{t_0-\delta}^{t_0+\delta} \left(|t-t_0|+n^{-1}\right)^{\eta/2-\theta} dt \le C.$$
But,
$$\|p_n(w)u\|_p \sim \left\| \frac{u}{\sqrt{w\varphi}} \right\|_p$$
and $\eta/2-\theta > -1$.

To prove the second part of Theorem 3.1 it is sufficient to show that

$$\Gamma_n = \max_{x\in A} u(x) \sum_{k=1,k\neq d}^{n} \frac{|l_k(x)|}{u_n\,(x_k)} \le \mathcal{C}\log n$$

if and only if (3.11) holds. But, with $w_n(x) = v^{\alpha,\beta}(x)\left(|x-t_0| + n^{-1}\right)^{\eta}$ and $x \in A$

$$\begin{aligned}\Gamma(x) \;&:=\; |p_n(w,x)u(x)| \sum_{k=1,k\neq d}^{n} \frac{|l_k(x)|}{u_n\,(x_k)}\\ &\sim\; |p_n(w,x)\sqrt{w_n\varphi}(x)| \left(\frac{u_n(x)}{\sqrt{w\varphi}(x)} \sum_{k=1,k\neq d}^{n} \frac{\sqrt{w_n\varphi}\,(x_k)}{u_n\,(x_k)} \frac{\Delta x_k}{|x-x_k|} \right).\end{aligned}$$

Using Lemma 5.1, with

$$\mu = \gamma - \frac{\alpha}{2} - \frac{1}{4}\,, \quad \nu = \delta - \frac{\beta}{2} - \frac{1}{4}\,, \quad \rho = \theta - \frac{\eta}{2}\,,$$

we deduce

$$\Gamma(x) \sim \mathcal{C}\,|p_n(w,x)\sqrt{w\varphi}(x)|\log n$$

if and only if (3.11) are satisfied. Then the second part of Theorem 3.1 follows by recalling that $\left\|\, p_n(w)\sqrt{w_n\varphi}\,\right\| \sim 1$. $\Box$

Proof of Corollary 3.1. Firstly, we prove inequality (3.8). We have

$$\|u[f - \tilde{L}_n(w,F)]\|_p \le \|u[f-F]\|_p + \|u[F - \tilde{L}_n(w,F)]\|_p.$$

By the definition of F, it follows

$$\|u[f-F]\|_p \le \mathcal{C} \inf_{q_s\in\mathcal{P}_s} \|u[f-q_s]\|_{L^\infty\left(t_0-\frac{\mathcal{C}}{n},\, t_0+\frac{\mathcal{C}}{n}\right)}.$$

Moreover, for all polynomials $P \in \mathcal{P}_{n-1}$, using Theorem 3.1 we get

$$\begin{aligned}\|u[F - \tilde{L}_n(w,F)]\|_p \;&\le\; \|u(F-P)\|_p + \|u\tilde{L}_n(w,F-P)\|_p\\ &\le\; \mathcal{C}\|u(F-P)\|_\infty \le \mathcal{C}\|u(F-f)\|_\infty + \|u(f-P)\|_\infty\\ &\le\; \mathcal{C} \inf_{q_s\in\mathcal{P}_s} \|u(f-q_s)\|_\infty + \|u(f-P)\|_\infty.\end{aligned}$$

Therefore, assuming that $P \in \mathcal{P}_{n-1}$ minimizes the last expression, we arrive at the estimate

$$\|u[f - \tilde{L}_n(w,F)]\|_p \le \mathcal{C}[E_{n-1}(f)_{u,\infty} + \inf_{q_s\in\mathcal{P}_s} \|u(f-q_s)\|_\infty],$$

i.e., (3.8).

The proof of (3.9) is similar and therefore Corollary 3.1 is proved. □

Proof of Theorem 3.3. Let $d = \max(t_0 - y_1, y_r - t_0)$ and set

$$A = \left(-1 + \frac{a}{n^2}, t_0 - 2d\right) \cup \left(t_0 + 2d, 1 - \frac{a}{n^2}\right),$$

where $a > 0$ is fixed. Since the measure of $[t_0 - 2d, t_0 + 2d]$ is of order n^{-1}, we use the Remez inequality to obtain

$$\|uL_n(v^{\alpha,\beta}, f)\|_p \le \mathcal{C}\|uL_n(v^{\alpha,\beta}, f)\|_{L^p(A)}, \quad 1 \le p \le +\infty.$$

Moreover, for $x \in A$, i.e., $|t_0 - x| > 2d$ and $\frac{1}{n} \le \mathcal{C}|x - t_0|$, it results

$$\frac{|x - t_0|}{2} < |x - y_i| \le \mathcal{C}\frac{|x - t_0|}{2}$$

and

$$\pi(x) \sim |x - t_0|^\nu, \quad x \in A.$$

Then, letting $q_n(x) = P_{n+\nu}(v^{\alpha,\beta}, x)/\pi(x)$,

$$g(x) = \operatorname{sgn} L_n(v^{\alpha,\beta}, f, x)|u(x)L_n(v^{\alpha,\beta}, f, x)|^{p-1}$$

and

$$r(t) = \int_A \frac{q_n(x) - q_n(t)}{x - t} u(x)g(x)dx \in \mathcal{P}_{n-1},$$

we can write

$$\|uL_n(v^{\alpha,\beta}, f)\|_{L^p(A)} \le \mathcal{C} \sum_{\substack{k=1 \\ x_k \notin \mathcal{B}}}^n \frac{|f(x_k)\pi(x_k)|}{P'_{n+\nu}(v^{\alpha,\beta}, x_k)}|r(x_k)|.$$

Recalling the relation

$$\frac{1}{|P'_{n+\nu}(v^{\alpha,\beta}, x_k)|} \sim \sqrt{v^{\alpha,\beta}\varphi}(x_k)(x_{k+1} - x_k),$$

we get

$$\|uL_n(v^{\alpha,\beta}, f)\|_p \le \mathcal{C}\|uf\|_\infty \sum_{\substack{k=1 \\ x_k \notin \mathcal{B}}}^n \frac{\sqrt{v^{\alpha,\beta}\varphi}(x_k)}{|x_k - t_0|^{\theta-\nu}}|r(x_k)|\Delta x_k, \ 1 \le p < +\infty.$$

Now, if we repeat step by step the proof of Theorem 3.1 and recalling that

$$|q_n(x)| \le \frac{\mathcal{C}}{|x - t_0|^\nu \sqrt{v^{\alpha,\beta}\varphi(x)}}, \quad x \in A,$$

the equivalence between (3.16) and (3.17) follows easily.

Finally, we consider the case $p = +\infty$. Let

$$l_k^*(x) = \frac{P_{n+\nu}(v^{\alpha,\beta}, x)}{\pi(x)} \frac{\pi(x_k)}{P'_{n+\nu}(v^{\alpha,\beta}, x_k)(x - x_k)}.$$

Then, denoting by x_d one of the closest zeros to x, we have

$$u(x)\frac{|l_d^*(x)|}{u(x_d)} \sim 1,$$

$$u(x)\frac{|l_k^*(x)|}{u(x_k)} \sim \frac{|u(x)P_{n+\nu}(v^{\alpha,\beta}, x)|}{|x - t_0|^\nu} \frac{\Delta x_k}{v^{\gamma-\frac{\alpha}{2}-\frac{1}{4},\delta-\frac{\beta}{2}-\frac{1}{4}}(x_k)|t_0 - x_k|^{\theta-\nu}|x - x_k|},$$

which implies

$$\max_{x\in A} \sum_{\substack{k=1\\ x_k\notin\mathcal{B}}}^n u(x)\frac{|l_k^*(x)|}{u(x_k)} \sim 1 + \frac{|u(x)P_{n+\nu}(v^{\alpha,\beta}, x)|}{|x - t_0|^\nu|x - \tau|^{\theta-\nu}v^{\sigma,\tau}(x)} \times$$

$$\times \sum_{\substack{k=1\\ x_k\notin\mathcal{B}}}^n \frac{|x - \tau|^{\theta-\nu}v^{\sigma,\tau}(x)\Delta x_k}{|x_k - \tau|^{\theta-\nu}v^{\sigma,\tau}(x_k)|x - x_k|},$$

where

$$k \neq d, \quad \sigma = \gamma - \frac{\alpha}{2} - \frac{1}{4}, \quad \tau = \delta - \frac{\beta}{2} - \frac{1}{4}.$$

Moreover,

$$\frac{|u(x)P_{n+\nu}(v^{\alpha,\beta}, x)|}{|x - t_0|^\nu|x - \tau|^{\theta-\nu}v^{\sigma,\tau}(x)} = |v^{\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}}(x)P_{n+\nu}(v^{\alpha,\beta}, x)|$$

and

$$\max_{x\in A} |v^{\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}}(x)P_{n+\nu}(v^{\alpha,\beta}, x)| \sim 1.$$

Then, using Lemma 5.1, we have

$$\sup_{\|uf\|_\infty=1} \|uL_n(v^{\alpha,\beta}, f)\|_\infty \sim \max_{x\in A} \sum_{\substack{k=1\\ x_k\notin\mathcal{B}}}^n u(x)\frac{|l_k^*(x)|}{u(x_k)} \sim \log n,$$

if and only if (3.19) holds. □

Proof of Theorem 3.4. We first prove that

$$\sup_{\|vf_j\|=1} \|vL_{n+1}(w_\alpha, f_j)\|_\infty \sim \log n \tag{5.3}$$

holds if and only if

$$\frac{\alpha}{2} + \frac{1}{4} \leq \gamma \leq \frac{\alpha}{2} + \frac{5}{4} \quad \text{and} \quad \eta - 1 \leq \nu \leq \eta.$$

To this end we set $d := \max(t_0 - y_1, y_\nu - t_0)$ and

$$A = \left(\frac{\mathcal{C}}{n}, t_0 - 2d\right) \cup (t_0 + 2d, 4n).$$

Since the distance between the zeros in a neighborhood of t_0 is of order $1/\sqrt{n}$ (see [5]), we can use a Remez-type inequality [7] to obtain

$$\|vL_{n+1}(w_\alpha, f_j)\|_\infty \sim \|vL_{n+1}(w_\alpha, f_j)\|_{L^\infty(A)}$$

and

$$|\pi(x)| \sim |x - t_0|^\nu, \quad x \in A.$$

Moreover (cf. [5]), by easy computations we get

$$v(x)\frac{|l_d^*(x)|}{v(x_d)} \sim 1$$

and, for $j \geq k \neq d$,

$$\begin{aligned} v(x)\frac{|l_k^*(x)|}{v(x_k)} \quad &\sim \quad \left|\sqrt{w_\alpha(x)}P_{n+\nu}(w_\alpha, x)\right| \sqrt[4]{x(4n-x)} \;\times \\ &\qquad \times \left(\frac{x - t_0}{x_k - t_0}\right)^{\eta-\nu} \left(\frac{x}{x_k}\right)^{\gamma - \frac{\alpha}{2} - \frac{1}{4}} \frac{\Delta x_k}{|x - x_k|}. \end{aligned}$$

Since

$$\max_{x\in A} |\sqrt{w_\alpha(x)}P_{n+\nu}(w_\alpha, x)| \sqrt[4]{x(4n-x)}| \sim 1,$$

we conclude that

$$\begin{aligned} \sup_{\|f_j v\|_\infty = 1} \|L_{n+1}(w_\alpha, f)v\|_\infty \quad &\sim \quad \sup_{\|f_j v\|_\infty = 1} \|L_{n+1}(w_\alpha, f_j)v\|_{L^\infty(A)} \\ &\sim \quad \max_{x \in A} \sum_{\substack{k=1 \\ x_k \notin \mathcal{B}}}^{j} v(x)\frac{|l_k^*(x)|}{v(x_k)} \\ &\sim \quad 1 + \sum_{\substack{k=1 \\ x_k \notin \mathcal{B}}}^{j} \left|\frac{x - t_0}{x_k - t_0}\right|^{\eta-\nu} \left(\frac{x}{x_k}\right)^{\gamma - \frac{\alpha}{2} - \frac{1}{4}} \frac{\Delta x_k}{|x - x_k|}. \end{aligned}$$

By Lemma 5.1, the last sum is equivalent to $\log n$ if and only if

$$\frac{\alpha}{2} + \frac{1}{4} \leq \gamma \leq \frac{\alpha}{2} + \frac{5}{4} \quad \text{and} \quad \eta - 1 \leq \nu \leq \eta.$$

Now, we have

$$\|v[f - L_{n+1}(w_\alpha, f_j)]\|_\infty \leq \|v[f - f_j]\|_\infty + \|v[f_j - L_n(w_\alpha, f_j)]\|_\infty.$$

Letting

$$M = \left[\frac{\theta}{1+\theta}(n+\nu)\right] \sim n,$$

we have (see [6])

$$\|v[f-f_j]\|_\infty \le \mathcal{C}\left(E_M(f)_v + e^{-An}\|vf\|_\infty\right).$$

Moreover, since for all polynomials $P \in \mathcal{P}_M$, $P = P_j + \psi_j P$, and

$$\begin{aligned} f_j - L_n(w_\alpha, f_j) &= f_j - P - L_n(w_\alpha, f_j - P_j) + L_n(w_\alpha, \psi_j P) \\ &= (f_j - f) + (f - P) - L_n(w_\alpha, (f-P)_j) + L_n(w_\alpha, \psi_j P), \end{aligned}$$

we have

$$\begin{aligned} \|v[f_j - L_n(w_\alpha, f_j)]\|_\infty &\le \|v(f-P)\|_\infty + \|v(f-f_j)\|_\infty \\ &+ \|vL_n(w_\alpha,(f-P)_j)\|_\infty + \|vL_n(w_\alpha,\psi_j P)\|_\infty. \end{aligned}$$

Taking the infimum over $P \in \mathcal{P}_M$ and using (5.3), we see that the first three terms are dominated by

$$\mathcal{C}\left(E_M(f)_v \log n + e^{-An}\|vf\|_\infty\right).$$

Now, it remains to estimate the last term.

Thus,

$$\begin{aligned} |v(x)L_n(w_\alpha, \psi_j P, x)| &= \left|\sum_{k>j} v(x)\frac{|l_k^*(x)|}{v(x_k)} P(x_k)v(x_k)\right| \\ &\le \|vP\|_{[4\theta n, 4n]} \sum_{\substack{k>j \\ x_k \notin \mathcal{B}}} v(x)\frac{|l_k^*(x)|}{v(x_k)}. \end{aligned}$$

Using Lemma 5.1 and recalling the conditions on $\alpha, \beta, \gamma, \delta, \nu$, and η, we see that the last sum is of order $\log n$. Finally, using an inequality proved in [7], we obtain

$$\|vP\|_{[4\theta n,\infty)} \le \mathcal{C}e^{-An}\|vP\|_\infty \le \mathcal{C}e^{-An}\|vf\|_\infty,$$

since P is the polynomial of best approximation of $f \in L_v^\infty$. □

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